

Weak-Coupling Theory for Multiband Superconductivity Induced by Jahn-Teller Phonons

Takashi HOTTA

Department of Physics, Tokyo Metropolitan University, 1-1 Minami-Osawa, Hachioji, Tokyo 192-0397, Japan

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Emergence of superconductivity in a two-band system coupled with breathing and Jahn-Teller phonons is discussed in a weak-coupling limit. With the use of a standard quantum mechanical procedure, the phonon-mediated attraction is derived. From the analysis of the model including such attraction, a BCS-like formula for a superconducting transition temperature T_c is obtained. When only the breathing phonon is considered, T_c is the same as that of the one-band model. On the other hand, when Jahn-Teller phonons are active, T_c is significantly enhanced by the interband attraction even within the weak-coupling limit. Relevance of the present result to actual materials such as iron pnictides is briefly commented.

KEYWORDS: Jahn-Teller phonon, multiband, superconductivity

After the Bardeen-Cooper-Schrieffer (BCS) theory for superconductivity,¹⁾ it has been pointed out that anisotropic Cooper pairs can be formed by the Friedel oscillation in electron systems, mainly from an academic viewpoint.²⁾ Nowadays it is widely recognized that anisotropic superconductivity originating from strong electron correlation confirms an important route to achieve high superconducting transition temperature T_c . In fact, due to successive discoveries of superconductivity in strongly correlated electron systems such as molecular conductors, transition metal oxides, and heavy-fermion compounds, it has been one of central issues in the research field of condensed matter physics to elucidate the mechanism of anisotropic superconductivity with relatively high T_c . Among them, concerning a superconducting material group characterized by singlet Cooper pair, a key concept of d -wave superconductivity mediated by antiferromagnetic spin fluctuations has been believed to be established.³⁻⁸⁾

In addition to the concept of anisotropic superconductivity mediated by magnetic fluctuations, another important ingredient is multiband effect, since superconductivity has been found in electron systems with multiband such as Sr_2RuO_4 ,⁹⁾ MgB_2 ,¹⁰⁾ and iron pnictides.^{11,12)} Since multi-sheets of Fermi surfaces have been usually observed in heavy fermion compounds, the occurrence of superconductivity in such materials should be also related to multiband nature. Just after the BCS theory, multiband effect on T_c has been discussed in a simple two-band electron model.¹³⁾ In fact, in recent years, multiband superconductivity has been actively investigated from various viewpoints.¹⁴⁻²²⁾ In particular, here we mention s_{\pm} -wave superconductivity proposed for iron pnictides.²³⁻²⁶⁾

In general, in electron systems with degenerate orbitals, Jahn-Teller phonons should play an important role, since Jahn-Teller distortions are known to lift the degeneracy in electron orbitals. In fullerene superconductors,²⁷⁻³⁰⁾ s -wave pair formation due to Jahn-Teller phonons has been discussed. A possibility of superconductivity due to geometric phase in Jahn-Teller crystals has been proposed.^{31,32)} However, the attractive interaction mediated by Jahn-Teller phonons has not been analyzed satisfactorily even in a weak-coupling limit, probably because of tedious calculations to derive such effective attractions in multiband systems.

In this paper, the effective interaction is derived in the two-orbital electron system which is coupled with breathing and

Jahn-Teller phonons with the use of a standard quantum mechanical techniques. By applying a weak-coupling approximation to the effective model, we discuss the enhancement of T_c due to Jahn-Teller phonons in the two-band electron system. When we include only the breathing phonon, T_c is just the same as that of the famous BCS formula for the one-band model. On the other hand, when we consider Jahn-Teller phonons, we find the significant enhancement of T_c by the interband attraction. These weak-coupling solutions are checked from the numerical estimation of pair susceptibility in the effective 4-site model. It is concluded that T_c of the multiband systems coupled with phonons is enhanced by the number of relevant phonon modes.

Now we consider a two-orbital electron system coupled with breathing and Jahn-Teller phonons, given by

$$H = \sum_{\mathbf{ia}, \gamma\gamma'\sigma} t_{\gamma\gamma'}^{\mathbf{a}} d_{\mathbf{i}\gamma\sigma}^{\dagger} d_{\mathbf{i}+\mathbf{a}\gamma'\sigma} + \sum_{\ell, \mathbf{i}} g_{\ell} Q_{\ell\mathbf{i}} \rho_{\ell\mathbf{i}} + \sum_{\ell, \mathbf{i}} [P_{\ell\mathbf{i}}^2 / (2M_{\ell}) + k_{\ell} Q_{\ell\mathbf{i}}^2 / 2], \quad (1)$$

where $d_{\mathbf{i}\gamma\sigma}$ is an annihilation operator for an electron with spin σ in the orbital γ ($=a$ and b) at site \mathbf{i} , $t_{\gamma\gamma'}^{\mathbf{a}}$ denotes the electron hopping between adjacent γ - and γ' -orbitals in nearest neighbor sites connected by a vector \mathbf{a} , g_{ℓ} indicates the coupling constant between electrons and local distortion specified by ℓ , $Q_{1\mathbf{i}}$ indicates breathing distortion, $Q_{2\mathbf{i}}$ and $Q_{3\mathbf{i}}$ denote Jahn-Teller distortions, $\rho_{1\mathbf{i}} = d_{\mathbf{ia}\sigma}^{\dagger} d_{\mathbf{ia}\sigma} + d_{\mathbf{ib}\sigma}^{\dagger} d_{\mathbf{ib}\sigma}$, $\rho_{2\mathbf{i}} = d_{\mathbf{ia}\sigma}^{\dagger} d_{\mathbf{ib}\sigma} + d_{\mathbf{ib}\sigma}^{\dagger} d_{\mathbf{ia}\sigma}$, $\rho_{3\mathbf{i}} = d_{\mathbf{ia}\sigma}^{\dagger} d_{\mathbf{ia}\sigma} - d_{\mathbf{ib}\sigma}^{\dagger} d_{\mathbf{ib}\sigma}$, $P_{\ell\mathbf{i}}$ denotes canonical momentum of $Q_{\ell\mathbf{i}}$, M_{ℓ} is corresponding reduced mass, and k_{ℓ} denotes the spring constant.

By following the standard procedure of quantization of phonons, we introduce the phonon operator $a_{\ell\mathbf{i}}$, defined through $Q_{\ell\mathbf{i}} = (a_{\ell\mathbf{i}} + a_{\ell\mathbf{i}}^{\dagger}) / \sqrt{2\omega_{\ell} M_{\ell}}$, where ω_{ℓ} is the phonon energy given by $\omega_{\ell} = \sqrt{k_{\ell} / M_{\ell}}$. After performing the Fourier transform, we obtain H in the form of $H = H_0 + H_1$, where H_0 denotes the sum of electron and phonon energy, given by

$$H_0 = \sum_{\mathbf{k}, \tau\sigma} E_{\mathbf{k}\tau} d_{\mathbf{k}\tau\sigma}^{\dagger} d_{\mathbf{k}\tau\sigma} + \sum_{\ell, \mathbf{q}} \omega_{\ell} (a_{\ell\mathbf{q}}^{\dagger} a_{\ell\mathbf{q}} + 1/2). \quad (2)$$

Here the electron energy is given by

$$E_{\mathbf{k}\tau} = \left[\varepsilon_{\mathbf{k}aa} + \varepsilon_{\mathbf{k}bb} \pm \sqrt{(\varepsilon_{\mathbf{k}aa} - \varepsilon_{\mathbf{k}bb})^2 + 4\varepsilon_{\mathbf{k}ab}^2} \right] / 2, \quad (3)$$

where $\tau=1$ and 2 correspond to $+$ and $-$ signs, respectively, and $\varepsilon_{\mathbf{k}\gamma\gamma'} = \sum_{\mathbf{a}} e^{i\mathbf{k}\cdot\mathbf{a}} t_{\gamma\gamma'}^{\mathbf{a}}$. The electron-phonon coupling term H_1 is given by

$$H_1 = \sum_{\ell, \mathbf{k}, \mathbf{q}, \tau, \tau'} \sqrt{\alpha_{\ell}} \omega_{\ell} (a_{\ell \mathbf{q}} + a_{\ell -\mathbf{q}}^{\dagger}) \Gamma_{\tau\tau'}^{(\ell)}(\mathbf{k}, \mathbf{q}) \rho_{\tau\tau'}(\mathbf{k}, \mathbf{q}), \quad (4)$$

where $\alpha_{\ell} = g_{\ell}^2 / (2M_{\ell}\omega_{\ell}^3)$, $\rho_{\tau\tau'}(\mathbf{k}, \mathbf{q}) = \sum_{\sigma} d_{\mathbf{k}+\mathbf{q}\tau\sigma}^{\dagger} d_{\mathbf{k}\tau'\sigma}$, and the coefficient matrices $\hat{\Gamma}$'s are given by

$$\begin{aligned} \hat{\Gamma}^{(1)}(\mathbf{k}, \mathbf{q}) &= \begin{pmatrix} A_{\mathbf{k}, \mathbf{q}}^{+} & B_{\mathbf{k}, \mathbf{q}}^{-} \\ -B_{\mathbf{k}, \mathbf{q}}^{-} & A_{\mathbf{k}, \mathbf{q}}^{+} \end{pmatrix}, \\ \hat{\Gamma}^{(2)}(\mathbf{k}, \mathbf{q}) &= \begin{pmatrix} -B_{\mathbf{k}, \mathbf{q}}^{+} & A_{\mathbf{k}, \mathbf{q}}^{-} \\ A_{\mathbf{k}, \mathbf{q}}^{-} & B_{\mathbf{k}, \mathbf{q}}^{+} \end{pmatrix}, \\ \hat{\Gamma}^{(3)}(\mathbf{k}, \mathbf{q}) &= \begin{pmatrix} A_{\mathbf{k}, \mathbf{q}}^{-} & B_{\mathbf{k}, \mathbf{q}}^{+} \\ B_{\mathbf{k}, \mathbf{q}}^{+} & -A_{\mathbf{k}, \mathbf{q}}^{-} \end{pmatrix}. \end{aligned} \quad (5)$$

Here $A_{\mathbf{k}, \mathbf{q}}^{\pm} = u_{\mathbf{k}+\mathbf{q}}^{+} u_{\mathbf{k}}^{+} \pm u_{\mathbf{k}+\mathbf{q}}^{-} u_{\mathbf{k}}^{-}$ and $B_{\mathbf{k}, \mathbf{q}}^{\pm} = u_{\mathbf{k}+\mathbf{q}}^{+} u_{\mathbf{k}}^{-} \pm u_{\mathbf{k}+\mathbf{q}}^{-} u_{\mathbf{k}}^{+}$, where $u_{\mathbf{k}}^{\pm}$ is given by

$$u_{\mathbf{k}}^{\pm} = \sqrt{\frac{1}{2}} \left[1 \pm \frac{\varepsilon_{\mathbf{k}aa} - \varepsilon_{\mathbf{k}bb}}{\sqrt{(\varepsilon_{\mathbf{k}aa} - \varepsilon_{\mathbf{k}bb})^2 + 4\varepsilon_{\mathbf{k}ab}^2}} \right]^{1/2}. \quad (6)$$

Since we assume degenerate the Jahn-Teller modes here, we set $\omega_1 = \omega_{\text{br}}$ and $\omega_2 = \omega_3 = \omega_{\text{JT}}$. Concerning coupling constants, we introduce $\alpha_1 = \alpha_{\text{br}}$ and $\alpha_2 = \alpha_3 = \alpha_{\text{JT}}$.

Now we derive the effective Hamiltonian H_{eff} from H due to the elimination of phonon degrees of freedom by using a canonical transformation. Let us here consider a transformation $H_{\text{eff}} = e^S H e^{-S}$, where the operator S is defined so as to satisfy the relation $H_1 = -[H_0, S]$. Then, after some calculations of operators, we obtain the effective Hamiltonian as $H_{\text{eff}} = H_0 + H_{\text{int}}$. The effective interaction between electrons mediated by phonons is given by

$$H_{\text{int}} = -[[H_0, S], S]/2. \quad (7)$$

Here we obtain the effective model within the second order of electron-phonon coupling constant.

In the present case, first we assume S in the form of

$$\begin{aligned} S &= \sum_{\ell, \mathbf{k}, \mathbf{q}, \tau, \tau'} \sqrt{\alpha_{\ell}} \omega_{\ell} [X_{\tau\tau'}^{-(\ell)}(\mathbf{k}, \mathbf{q}) a_{\ell \mathbf{q}} \\ &\quad + X_{\tau\tau'}^{+(\ell)}(\mathbf{k}, \mathbf{q}) a_{\ell -\mathbf{q}}^{\dagger}] \rho_{\tau\tau'}(\mathbf{k}, \mathbf{q}), \end{aligned} \quad (8)$$

where $X_{\tau\tau'}^{\pm(\ell)}$ is determined so as to satisfy $H_1 = -[H_0, S]$. After lengthy calculations, we obtain

$$X_{\tau\tau'}^{\pm(\ell)}(\mathbf{k}, \mathbf{q}) = \frac{-\sqrt{\alpha_{\ell}} \omega_{\ell}}{E_{\mathbf{k}+\mathbf{q}\tau} - E_{\mathbf{k}\tau'} \pm \omega_{\ell}} \Gamma_{\tau\tau'}^{(\ell)}(\mathbf{k}, \mathbf{q}). \quad (9)$$

The effective interaction H_{int} is evaluated by $[H_1, S]/2$ from eq. (7). Then, we obtain

$$\begin{aligned} H_{\text{int}} &= \sum_{\ell, \tau, \tau', \mu, \mu'} \sum_{\mathbf{k}, \mathbf{k}', \mathbf{q}} U_{\ell} \frac{\omega_{\ell}^2 \Gamma_{\tau\tau'}^{(\ell)}(\mathbf{k}, \mathbf{q}) \Gamma_{\mu\mu'}^{(\ell)}(\mathbf{k}', -\mathbf{q})}{(E_{\mathbf{k}\tau'} - E_{\mathbf{k}'\tau})^2 - \omega_{\ell}^2} \\ &\quad \times \rho_{\tau\tau'}(\mathbf{k}, \mathbf{q}) \rho_{\mu\mu'}(\mathbf{k}', -\mathbf{q}), \end{aligned} \quad (10)$$

where $U_{\ell} = \alpha_{\ell} \omega_{\ell}$ and U_{ℓ} denotes the attractive interaction due to ℓ -mode phonons. Note here that $U_1 = U_{\text{br}} = \alpha_{\text{br}} \omega_{\text{br}}$ and $U_2 = U_3 = U_{\text{JT}} = \alpha_{\text{JT}} \omega_{\text{JT}}$.

Here we provide a comment on the Cooper pair in the sys-

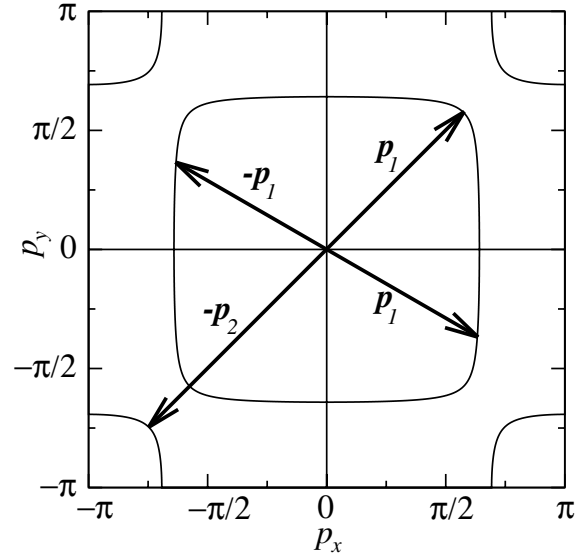


Fig. 1. Cooper pairs in a two-dimensional electron system with a couple of Fermi surfaces. In the weak-coupling limit, pairs are formed by electrons on the same Fermi surface, since the pair between different Fermi surfaces should be composed of electrons with \mathbf{p}_1 and $-\mathbf{p}_2$.

tem with multi-Fermi surfaces. In the weak-coupling limit, we usually consider the pairing of electrons only in the vicinity of the Fermi energy E_F . Thus, we set $E_{\mathbf{k}'-\mathbf{q}\tau} = E_{\mathbf{k}\tau'} \approx E_F$. As shown in Fig. 1, in the weak-coupling limit, the Cooper pair is formed only by the electrons on the same Fermi surface, except for an unrealistic case in which a couple of Fermi-surface sheets are perfectly degenerate. Then, we consider the interaction for Cooper pair with zero total momentum. Note that in the strong-coupling region, it is possible to consider the pairs of electrons far from the Fermi surface, leading to a chance of pair formation between different Fermi surfaces.

After some algebraic calculations, we obtain the effective interaction in the form of

$$H_{\text{int}} = - \sum_{\mathbf{p}, \mathbf{p}', \mu, \nu} V_{\mu\nu}(\mathbf{p}, \mathbf{p}') d_{\mathbf{p}\mu\uparrow}^{\dagger} d_{-\mathbf{p}\mu\downarrow}^{\dagger} d_{-\mathbf{p}'\nu\downarrow} d_{\mathbf{p}'\nu\uparrow}. \quad (11)$$

The pair potential $V_{\mu\nu}$ is given by

$$\begin{aligned} V_{11}(\mathbf{p}, \mathbf{p}') &= V_{22}(\mathbf{p}, \mathbf{p}') = U_{\text{JT}} + U_{\text{br}} A_{\mathbf{p}', \mathbf{p}-\mathbf{p}'}^{+2}, \\ V_{12}(\mathbf{p}, \mathbf{p}') &= V_{21}(\mathbf{p}, \mathbf{p}') = U_{\text{JT}} + U_{\text{br}} B_{\mathbf{p}', \mathbf{p}-\mathbf{p}'}^{-2}. \end{aligned} \quad (12)$$

Here we note the relation of $A_{\mathbf{p}', \mathbf{p}-\mathbf{p}'}^{+2} + B_{\mathbf{p}', \mathbf{p}-\mathbf{p}'}^{-2} = 1$.

Next we solve the gap equation. The gap function $\Delta_{\mu}(\mathbf{p})$ is given by

$$\Delta_{\mu}(\mathbf{p}) = \sum_{\mathbf{p}', \nu} V_{\mu\nu}(\mathbf{p}, \mathbf{p}') \langle d_{-\mathbf{p}'\nu\downarrow} d_{\mathbf{p}'\nu\uparrow} \rangle, \quad (13)$$

where $\langle \cdots \rangle$ denotes the average by using H in the mean-field approximation. Note again that it is enough to consider the pairs on the same Fermi surfaces in the weak-coupling limit. By assuming the Cooper pair with s -wave symmetry, we obtain the gap equation at $T = T_c$ as

$$\Delta_{\mu} = \log(1.13\omega_c/T_c) \sum_{\nu} \lambda_{\mu\nu} \Delta_{\nu}, \quad (14)$$

where ω_c is an appropriate cut-off frequency and $\lambda_{\mu\nu}$ is the

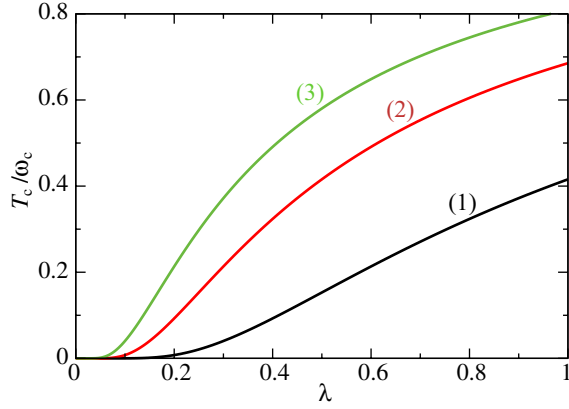


Fig. 2. (Color online) Curves of (1) $T_c/\omega_c = 1.13e^{-1/\lambda_{\text{br}}}$, (2) $T_c/\omega_c = 1.13e^{-1/(2\lambda_{\text{JT}})}$, and (3) $T_c/\omega_c = 1.13e^{-1/(2\lambda_{\text{JT}}+\lambda_{\text{br}})}$. Here we set $\lambda_{\text{JT}} = \lambda_{\text{br}} = \lambda$ for simplicity.

non-dimensional coupling constant, given by

$$\begin{aligned}\lambda_{11} &= \lambda_{22} = \lambda_{\text{JT}} + \lambda_{\text{br}}\beta, \\ \lambda_{12} &= \lambda_{21} = \lambda_{\text{JT}} + \lambda_{\text{br}}(1 - \beta).\end{aligned}\quad (15)$$

Here $\lambda_{\text{JT}} = N_0 U_{\text{JT}}$, $\lambda_{\text{br}} = N_0 U_{\text{br}}$, N_0 denotes the density of states at the Fermi level, $\beta = \langle A_{\mathbf{p}, \mathbf{p}'}^{+2} \rangle_{\text{FS}}$, and $\langle \dots \rangle_{\text{FS}}$ denotes the average over the Fermi surface. Note that we simply assume the same values of N_0 for the different Fermi surfaces.

By solving the gap equation eq. (14), we obtain

$$T_c = 1.13\omega_c e^{-1/(\lambda_{11}+\lambda_{12})} = 1.13\omega_c e^{-1/(2\lambda_{\text{JT}}+\lambda_{\text{br}})}. \quad (16)$$

Note that another solution provides smaller T_c even if it exists. Equation (16) tells us several interesting stories. First we consider a situation in which only the breathing mode is active. In this case, we immediately obtain the same formula of T_c , $T_c = 1.13\omega_c e^{-1/\lambda_{\text{br}}}$, as that of the BCS theory for a one-band system. Namely, even if the number of the band is increased, the magnitude of T_c is not changed as long as the total electron density is coupled with the breathing phonons. In this situation, there is no advantageous points of multi-band nature for the elevation of T_c .

On the other hand, in the multi-orbital system, there occurs a coupling with Jahn-Teller phonons so as to lift the degeneracy in electron systems. In such a case, the factor 2 appears in front of the coupling constant in the T_c formula. In other words, this factor 2 indicates the number of Jahn-Teller modes, *not* the number of electron bands. When we consider the coupling of degenerate electrons with both breathing and Jahn-Teller phonons, the factor 3 becomes effective if we simply consider $\lambda_{\text{br}} = \lambda_{\text{JT}}$. In Fig. 2, we show the curves of T_c for the three cases of (1) breathing phonon, (2) Jahn-Teller phonons, and (3) both breathing and Jahn-Teller phonons. Here for simplicity, we set $\lambda_{\text{br}} = \lambda_{\text{JT}} = \lambda$. As easily understood, the change of the factor in the power is remarkable, even in the weak-coupling approximation. Note that the factor 2 or 3 indicates the total number of phonon modes which are coupled with electron systems.

In order to confirm the present result of eq. (16) obtained in the weak-coupling BCS approximation, we evaluate the pair susceptibility of the effective two-band model with the attractive interaction induced by phonons. For the purpose, we resort to an unbiased technique such as exact diagonalization.

The singlet pair correlation function is evaluated in a small-sized cluster and the effective coupling constant λ_{eff} is deduced from the singlet pair correlation.

The effective model is given by

$$\begin{aligned}H_{\text{eff}} &= \sum_{\mathbf{k}, \tau\sigma} E_{\mathbf{k}\tau} d_{\mathbf{k}\tau\sigma}^\dagger d_{\mathbf{k}\tau\sigma} - I \sum_{i, \tau} d_{i\tau\uparrow}^\dagger d_{i\tau\uparrow} d_{i\tau\downarrow}^\dagger d_{i\tau\downarrow} \\ &\quad - J \sum_i (d_{i1\uparrow}^\dagger d_{i1\downarrow}^\dagger d_{i2\downarrow} d_{i2\uparrow} + \text{h.c.}),\end{aligned}\quad (17)$$

where I and J denote on-site and pair-hopping *attractive* interactions corresponding to V_{11} and V_{12} , respectively. In order to reproduce the situation in eq. (15), we set I and J as $I = U_{\text{JT}} + U_{\text{br}}\beta$ and $J = U_{\text{JT}} + U_{\text{br}}(1 - \beta)$, respectively, by taking β as a parameter.

The pair susceptibility matrix $\chi_{\mu\nu}(\mathbf{m}, \mathbf{n})$ is defined by

$$\chi_{\mu\nu}(\mathbf{m}, \mathbf{n}) = \int_0^{1/T} d\tau \langle \hat{\phi}_{\mathbf{m}, \mu}(\tau) \hat{\phi}_{\mathbf{n}, \nu}^\dagger \rangle, \quad (18)$$

where T is a temperature, $\hat{\phi}_{\mathbf{m}, \mu}(\tau) = e^{H\tau} \hat{\phi}_{\mathbf{m}, \mu} e^{-H\tau}$, \mathbf{m} indicates the vector connecting possible two sites in the cluster, and $\hat{\phi}_{\mathbf{m}, \mu}$ is a singlet pair operator of the band μ , given by

$$\hat{\phi}_{\mathbf{m}, \mu} = \sum_i \phi_{\mathbf{m}, \mu} (d_{i\mu\downarrow} d_{i+\mathbf{m}\mu\uparrow} - d_{i\mu\uparrow} d_{i+\mathbf{m}\mu\downarrow}) \sqrt{2}. \quad (19)$$

The coefficient $\phi_{\mathbf{m}, \mu}$ is determined by the diagonalization of the susceptibility matrix and the pair susceptibility $\chi_{\mu\nu}$ is defined by its maximum eigenvalue.

In order to extract information on the pairing interaction, we consider the pair susceptibility in a diagrammatic manner. When we define the non-interacting pair susceptibility as $\hat{\chi}^{(0)}$, $\hat{\chi}$ satisfies the relation of $\hat{\chi} = \hat{\chi}^{(0)} + \hat{\chi}^{(0)} \hat{K} \hat{\chi}$ in the ladder approximation, where \hat{K} denotes the effective pairing interaction in the matrix form. Usually we obtain $\hat{\chi}$ from \hat{K} and $\hat{\chi}^{(0)}$, but here we evaluate \hat{K} as $\hat{K} = \hat{\chi}^{(0)-1} - \hat{\chi}^{-1}$, where $\hat{\chi}$ is numerically evaluated. Then, we can evaluate the effective non-dimensional coupling constant λ_{eff} as

$$\lambda_{\text{eff}} = (K_{11} + K_{12})/W, \quad (20)$$

where W is the bandwidth.

In this paper, for the evaluation of $\hat{\chi}$, we exploit an exact diagonalization technique for the model in a 4-site cluster. We set $E_{\mathbf{k}1} = -t(\cos k_x + \cos k_y)$ and $E_{\mathbf{k}2} = t(\cos k_x + \cos k_y)$, leading to a couple of Fermi surfaces around Γ and M points in the thermodynamic limit. Here we consider the case of $n=0.5$, where n is the electron number per site and per orbital. In this case, $\chi_{11}^{(0)} = \chi_{22}^{(0)} = 4/t$ and $W = 4t$ in the 4-site cluster.

In Fig. 3(a), we show λ_{eff} vs. β by solid symbols for $U_{\text{br}} = 0.05t, 0.1t, 0.2t$, and $0.3t$ with $n=0.5$, $U_{\text{JT}}=0$, and $T=0$. Note that each horizontal line denotes the pair susceptibility of a one-band model with nearest neighbor hopping t and on-site attraction U_{br} for the same value of n in the 4-site cluster. For small values of U_{br}/t , we find that λ_{eff} weakly depends on β and it agrees well with the result of the one-band model, suggesting that T_c in the two-band system coupled with breathing phonons is just the same as that of the one-band model. Namely, in such a case, T_c is not expected to increase even if we increase the number of electron bands.

In Fig. 3(b), we show λ_{eff} vs. U/t with a fixed value of $\beta=0.5$ for three cases as (1) $U_{\text{br}}=U$ and $U_{\text{JT}}=0$, (2) $U_{\text{br}}=0$ and $U_{\text{JT}}=U$, and (3) $U_{\text{br}}=U_{\text{JT}}=U$. The effective coupling constant

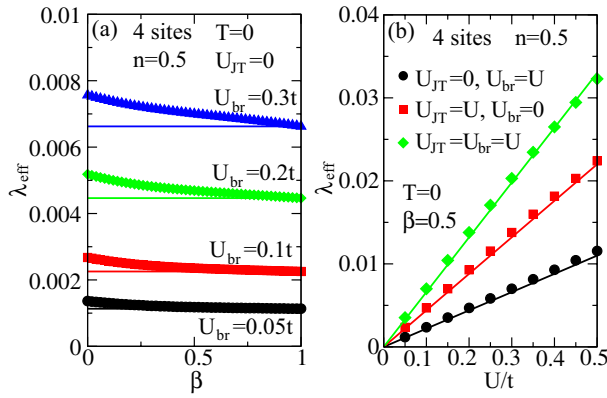


Fig. 3. (Color online) (a) Effective coupling constant λ_{eff} vs. β for $U_{\text{br}}/t=0.05, 0.1, 0.2$, and 0.3 with $n=0.5$, $U_{\text{JT}}=0$, and $T=0$. As for the horizontal lines, see the main text. (b) λ_{eff} vs. U/t for $(U_{\text{br}}, U_{\text{JT}})=(0,0)$, $(0,U)$ and (U,U) with $\beta=0.5$. Three lines indicate $\lambda_{\text{eff}} = 0.022 \times U/t$, $0.044 \times U/t$, and $0.066 \times U/t$.

is found to be in proportion to U/t . Among the proportional coefficients, we confirm the relation of $\lambda_{\text{eff}}^{(1)} = \lambda_{\text{eff}}^{(2)}/2 = \lambda_{\text{eff}}^{(3)}/3$, which is consistent with the weak-coupling result of eq. (16).

Thus far we have analyzed the models in the weak-coupling region, but we are also interested in the strong-coupling behavior. For instance, when we increase U_{br} in Fig. 3(a), β -dependence of λ_{eff} becomes more significant and the deviation from the one-band result is large. When we increase the value of attractive interaction in Fig. 3(b), λ_{eff} is no longer in proportion to U/t and the relation of $\lambda_{\text{eff}}^{(1)} = \lambda_{\text{eff}}^{(2)}/2 = \lambda_{\text{eff}}^{(3)}/3$ does not hold. In order to discuss such strong-coupling effects, we should analyze directly the original Hamiltonian eq. (1), not the effective model eq. (17), by applying the Migdal-Eliashberg theory. It is one of future problems.

Here we provide a brief comment on the polaron mass enhancement, which is one of strong coupling effects. For Holstein phonons, it has been well known that the polaron mass m^* is increased as $m^*/m = e^{\alpha_{\text{br}}}$, where m is the bare electron mass, while for Jahn-Teller phonons, the mass enhancement has been found to be expressed by $m^*/m \approx e^{\alpha_{\text{JT}}/2}/\sqrt{\alpha_{\text{JT}}}$ for large α_{JT} . Namely, for the same value of the coupling constant, the mass of the Jahn-Teller polaron is smaller than that of the Holstein polaron. This behavior seems to be related to the fact that the vertex corrections in an electron system coupled with Jahn-Teller phonons should be less effective in comparison with the case of Holstein phonons.^{33,34} The fact may be also relevant to the increase of superconducting T_c in electron systems coupled with Jahn-Teller phonons.

Note also that in the Migdal-Eliashberg theory, the effect of Coulomb interaction is included in the parameter μ^* as the reduced repulsion. The same story is expected to be applied to the present case, as long as we consider adiabatic phonons. However, the degree of the reduction of Coulomb repulsion may be different between intra- and inter-orbital interactions. If the on-site interaction is still negative while the pair hopping interaction becomes positive, we obtain the so-called s_{\pm} -wave pairing, proposed for iron pnictides.²³⁻²⁶ Thus, it may be possible to construct an alternative Jahn-Teller phononic scenario for superconductivity in iron pnictides. In particular,

competition between Coulomb repulsion and phonon-induced attraction may be a key issue to understand the appearance of both nodal d -wave and nodeless s_{\pm} -wave gaps in iron pnictides. This is an interesting problem in future.

In summary, we have discussed the appearance of superconductivity in the two-band electron system coupled with breathing and Jahn-Teller phonons in the weak-coupling limit. It has been found that T_c is increased with the increase of the number of relevant phonon modes. Namely, T_c of the two-band system coupled with Jahn-Teller phonons becomes high in comparison with that of the one-band case, leading to a possibility of phonon-induced high- T_c superconductivity.

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